MATHEMATICS

Application of derivatives

1. If a quantity y varies with another quantity x, satisfying some rule y = f(x), then $\frac{dx}{dy}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_o$.

2. If two variables x and y are varying with respect to another variable t, i.e., if x = f(t) and y =g(t) then by Chain Rule
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ if } \frac{dx}{dt} \neq 0$$

3. A function f is said to be increasing on an interval (a, b) if $x_1 < x_2$ in (a, b) ⇒ $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$. Alternatively, if $f'(x) > 0$ for each x in, then f(x) is an increasing function on (a, b).

4. A function f is said to be increasing on an interval (a, b) if $x_1 < x_2$ in (a, b) ⇒ $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$. Alternatively, if $f'(x) > 0$ for each x in, then f(x) is an decreasing function on (a, b).

5. The equation of the tangent at (x_0, y_0) to the curve y = f (x) is given by
$$y - y_o = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_o)$$

6. If $\frac{dy}{dx}$ does not exist at the point (x_0, y_0), then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.

7. If tangent to a curve y = f(x) at $x = x_0$ is parallel to x-axis, then $\left.\frac{dy}{dx}\right]_{x=x_0} = 0$

8. **Equation of the normal** to the curve y = f (x) at a point (x_o, y_o), is given by
$$y - y_o = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)$$

9. If $\frac{dy}{dx}$ at the point (x_o, y_o), is zero, then equation of the normal is $x = x_0$.

10. If $\frac{dy}{dx}$ at the point (x_o, y_o), does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$.

11. Let y = f (x), Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x, i.e., Δy = f (x + Δx) – f (x). Then dy given by dy = f′(x) dx or $\mathrm{dy} = \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)\mathrm{dx}$ is a good of Δy when dx x = Δ is relatively small and we denote it by dy ≈ Δy.

12. A point c in the domain of a function f at which either f ′(c) = 0 or f is not differentiable is called a critical point of f.

13. First Derivative Test: Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then,

i. If f ′(x) changes sign from positive to negative as x increases through c, i.e., if f ′(x) > 0 at every point sufficiently close to and to the left of c, and f ′(x) < 0 at every point sufficiently close to and to the right of c, then c is a point of local maxima.

ii. If f ′(x) changes sign from negative to positive as x increases through c, i.e., if f ′(x) < 0 at every point sufficiently close to and to the left of c, and f ′(x) > 0 at every point sufficiently close to and to the right of c, then c is a point of local minima.

iii. If f ′(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. In fact, such a point is called point of inflexion.

14. Second Derivative Test: Let f be a function defined on an interval I and c ∈ I. Let f be twice differentiable at c. Then,

i. x = c is a point of local maxima if f ′(c) = 0 and f ″(c) < 0

   The values f (c) is local maximum value of f.

ii. x = c is a point of local minima if f ′(c) = 0 and f ″(c) > 0

   In this case, f (c) is local minimum value of f.

iii. The test fails if f ′(c) = 0 and f ″(c) = 0.

   In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.

15. Working rule for finding absolute maxima and/ or absolute minima

Step 1: Find all critical points of f in the interval, i.e., find points x where either f ′(x) = 0 or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step3.

This maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

MIND MAP : LEARNING MADE SIMPLE CHAPTER - 6

Application of Derivatives

Rate of change of quantities

If a quantity if 'y' varies with another quantity x so that $y = f(x)$, then $\frac{dy}{dx}\left[f'(x)\right]$ represents the rate of change of y w.r.t x and $\left.\frac{dy}{dx}\right|_{x=x_o} \left(f'(x_0)\right)$ represents the rate of change of y w.r.t. x at $x = x_o$

If 'x' and 'y' varies with another variable 't' i.e., if $x = f(t)$ and $y = g(t)$, then by chain rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$.

For eg: if the radius of a circle, r = 5 cm, then the rate of change of the area of a circle per second w.r.t 'r' is – $\frac{da}{dr}|_{r=5} = \frac{d}{dr}(\pi r^2)|_{r=5} = 2\pi r|_{r=5} = 10\pi$

Increasing and decreasing functions

A function f is said to be (i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \forall x_1 x_2 \in (a,b)$, and (ii) decreasing on (a, b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) > f(x_2) \forall x_1 x_2 \in (a,b)$

If $f'(x) \geq 0 \forall x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \leq 0 \forall x \in (a,b)$, then f is decreasing in (a,b) For eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then $f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \forall x \in R$. So, the function f is strictly increasing on R.

Tangents and Normals

The equation of the tangent at (x_0, y_0), to the curve $y = f(x)$ is given by $(y - y_0) = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)$ if $\frac{dy}{dx}$ does not exists at (x_0, y_0), then the tangent at (x_0, y_0) is parallel to the y–axis and its equation is $x = x_0$. If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to x–axis, then $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$.

Equation of the normal to the curve

$y = f(x)$ at (x_0, y_0) is $y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right|_{(x_0,y_0)}} (x - x_0)$ if $\frac{dy}{dx}$ at (x_0, y_0) is zero, then equation of the normal is $x = x_0$. If $\frac{dy}{dx}$ at (x_0, y_0) does not exist, then the normal is parallel to x-axis and its equation is $y = y_0$ For eg: Let $y = x^3 - x$ be a curve, then the slope of the tangent to $y = x^3 - x$ at $x = 2$ is $\left.\frac{dy}{dx}\right|_{x=2} = 3x^2 - 1 = 3.2^2 - 1 = 11$

Maxima and Minima

A point C in the domain of 'f' at which either $f'(c) = 0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open I. Then **(i)** if $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima. **(ii)** If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima. **(iii)** If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on I and C∈I, f is twice differentiable at C. Then
(i) $x = C$ is a point of local max. If $f'(C) = 0$ and $f''(C) < 0$, $f(C)$ is local max. of f.
(ii) $x = C$ is a point of local min if $f'(C) = 0$ and $f''(C) > 0$. $f(C)$ is local min of f. **(iii)** The test fails if $f'(C) = 0$ and $f''(C) = 0$

Approximations

Let $y = f(x)$ Δx be a small increment in 'x' and Δy be the small increment in y corresponding to the increment in 'x', i.e. $\Delta y = f(x + \Delta x) - f(x)$. Then, Δy is given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$, is a good approximation of Δy when $dx = \Delta x$ is relatively small and denote by $dy \approx \Delta y$. For eg: Let us approximate $\sqrt{36.6}$. To do this, we take $y = \sqrt{x}$, $x = 36$, $\Delta x = 0.6$ then

$$\Delta y = \sqrt{x + dx} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6 \Rightarrow \sqrt{36.6} = 6 + \bar{d}y$$

Now, dy is approximately Δy and is given by $\bar{d}y$

$$= \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05. \text{ So, } \sqrt{36.6} \approx 6 + 0.05 = 6.05.$$

Important Questions

Multiple Choice questions-

1. The rate of change of the area of a circle with respect to its radius r at r = 6 cm is:

(a) 10π

(b) 12π

(c) 8π

(d) 11π

2. The total revenue received from the sale of x units of a product is given by $R(x)=3x^2+36x+5$. The marginal revenue, when x = 15 is:

(a) 116

(b) 96

(c) 90

(d) 126.

3. The interval in which $y = x^2 e^{-x}$ is increasing with respect to x is:

(a) $(-\infty, \infty)$

(b) (-2,0)

(c) $(2, \infty)$

(d) (0, 2).

4. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at x = 0 is

(a) 3

(b) $\frac{1}{3}$

(c) -3

(d) $-\frac{1}{3}$

5. The line y = x + 1 is a tangent to the curve $y^2 = 4x$ at the point:

(a) (1, 2)

(b) (2, 1)

(c) (1, -2)

(d) (-1, 2).

6. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of f(3.02) is:

(a) 47.66

(b) 57.66

(c) 67.66

(d) 77.66.

7. The approximate change in the volume of a cube of side x meters caused by increasing the side by 3% is:

(a) $0.06\ x^3\ m^3$

(b) $0.6\ x^3\ m^3$

(c) $0.09\ x^3m^3$

(d) $0.9\ x^3\ m^3$

8. The point on the curve $x^2 = 2y$, which is nearest to the point (0, 5), is:

(a) $(2\sqrt{2}, 4)$

(b) $(2\sqrt{2}, 0)$

(c) (0, 0)

(d) (2, 2).

9. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(a) 0

(b) 1

(c) 3

(d) $\frac{1}{3}$

10. The maximum value of $[x (x - 1) + 1]^{1/3}$, $0 \le x \le 1$ is

(a) $(\frac{1}{3})^{1/3}$

(b) $\frac{1}{2}$

(c) 1

(d) 0

Very Short Questions:

1. For the curve $y = 5x- 2x^3$, if increases at the rate of 2 units/sec., find the rate of change of the slope of the curve when x = 3. (C.B.S.E. 2017)
2. Without using the derivative, show that the function f(x) = 7x – 3 is a strictly increasing function in R.
3. Show that function:

   $f(x) = 4x^3 - 18x^2 - 27x - 7$ is always increasing in R. (C.B.S.E. 2017)
4. Find the slope of the tangent to the curve:

   $x = at^2$, y = 2at t = 2.
5. Find the maximum and minimum values, if any, of the following functions without using derivatives:

   (i) $f(x) = (2x-1)^2 + 3$

   (ii) $f(x)= 16x^2 - 16x + 28$

   (iii) $f(x) = -|x+ 1| + 3$

   (iv) f(x) = sin 2x + 5

   (v) f(x) = sin (sin x).
6. A particle moves along the curve $x^2 = 2y$. At what point, ordinate increases at die same rate as abscissa increases? (C.B.S.E. Sample Paper 2019-20)

Long Questions:

1. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? (C.B.S.E. Outside Delhi 2019)

2. Find the angle of intersection of the curves $x^2 + y^2 = 4$ and $(x – 2)^2 + y^2 = 4$, at the point in the first quadrant (C.B.S.E. 2018 C)

3. Find the intervals in which the function: $f(x) = – 2x^3 – 9x^2 – 12x + 1$ is (i) Strictly increasing (ii) Strictly decreasing. (C.B.S.E. 2018 C)

4. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 meters. Find the dimensions of the window to admit maximum light through the whole opening. (C.B.S.E. 2018 C)

Assertion and Reason Questions:

1. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

Assertion(A): For each real 't', then exist a point C in $[t, t+\pi]$ such that $f'(C) = 0$

Reason (R): $f(t)=f(t+2\pi)$ for each real t

2. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

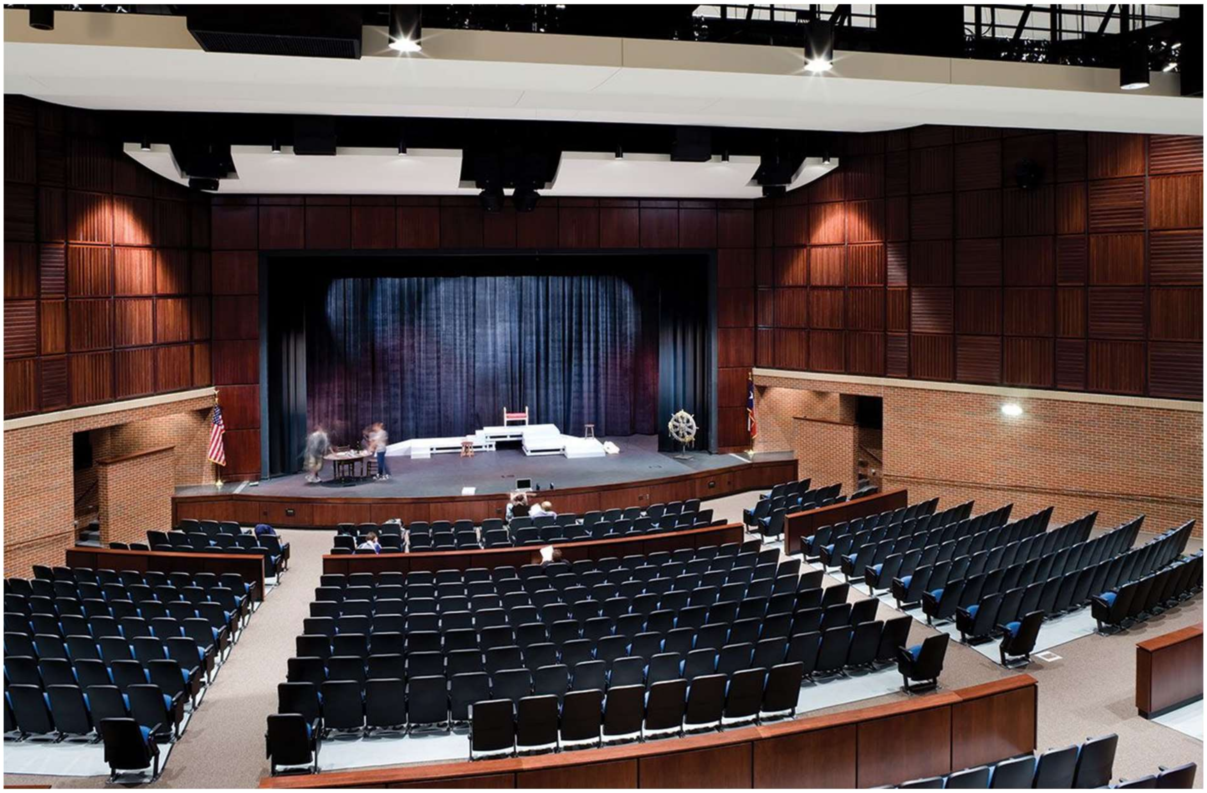
a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

Assertion (A): One root of $x^3−2x^2−1= 0$ and lies between 2 and 3.

Reason(R): If f(x) is continuous function and f [a], f[b] have opposite signs then at least one or odd number of roots of f(x)=0 lies between a and b.

Case Study Questions:

1. An architecture design a auditorium for a school for its cultural activities. The floor of the auditorium is rectangular in shape and has a fixed perimeter P.



Based on the above information, answer the following questions.

i. If x and y represents the length and breadth of the rectangular region, then relation between the variable is.

a. $x + y = P$
b. $x^2 + y^2 = P^2$
c. $2(x + y) = P$
d. $x + 2y = P$

ii. The area (A) of the rectangular region, as a function of x, can be expressed as.

a. $A = px + \frac{x}{2}$

b. $A = \frac{px + x^2}{2}$

c. $A = \frac{px - 2x^2}{2}$

d. $A = \frac{x^2}{2} + px^2$

iii. School's manager is interested in maximising the area of floor 'A' for this to be happen, the value of x should be.

a. P

b. $\frac{P}{2}$

c. $\frac{P}{3}$

d. $\frac{P}{4}$

iv. The value of y, for which the area of floor is maximum, is.

a. $\frac{P}{2}$

b. $\frac{P}{3}$

c. $\frac{P}{4}$

d. $\frac{P}{16}$

v. Maximum area of floor is.

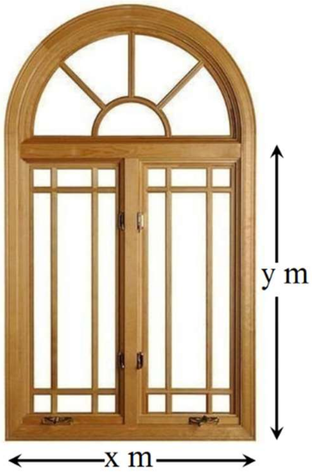
a. $\frac{P^2}{16}$

b. $\frac{P^2}{64}$

c. $\frac{P^2}{4}$

d. $\frac{P^2}{28}$

2. Rohan, a student of class XII, visited his uncle's flat with his father. He observe that the window of the house is in the form of a rectangle surmounted by a semicircular opening having perimeter 10m as shown in the figure.



Based on the above information, answer the following questions.

i. If x and y represents the length and breadth of the rectangular region, then relation between x and y can be represented as.

a. $x + y + \frac{\pi}{2} = 10$

b. $x + 2y + \frac{\pi x}{2} = 10$

c. $2x + 2y = 10$

d. $x + 2y + \frac{\pi}{2} = 10$

ii. The area (A) of the window can be given by.

a. $A = x - \frac{x^3}{8} - \frac{x^2}{2}$

b. $A = 5x - \frac{x^2}{8} - \frac{\pi x^2}{8}$

c. $A = x + \frac{\pi x^3}{8} - \frac{3x^2}{8}$

d. $A = 5x + \frac{x^3}{2} + \frac{\pi x^2}{8}$

iii. Rohan is interested in maximizing the area of the whole window, for this to happen, the value of x should be.

a. $\frac{10}{2-\pi}$

b. $\frac{20}{4-\pi}$

c. $\frac{20}{4+\pi}$

d. $\frac{10}{2+\pi}$

iv. Maximum area of the window is.

a. $\frac{30}{4+\pi}$

b. $\frac{30}{4-\pi}$

c. $\frac{50}{4-\pi}$

d. $\frac{50}{4+\pi}$

v. For maximum value of A, the breadth of rectangular part of the window is.

a. $\frac{10}{4+\pi}$

b. $\frac{10}{4-\pi}$

c. $\frac{20}{4+\pi}$

d. $\frac{20}{4-\pi}$

Answer Key-

Multiple Choice questions-

1. Answer: (b) 12π
2. Answer: (d) 126.
3. Answer: (d) (0, 2).

4. Answer: (d) $-\frac{1}{3}$
5. Answer: (a) (1, 2)
6. Answer: (d) 77.66.
7. Answer: (c) 0.09 x^3m^3
8. Answer: (a) (2 √2, 4)
9. Answer: (d) $\frac{1}{3}$
10. Answer: (c) 1

Very Short Answer:

1. Solution:

The given curve is $y = 5x - 2x^3$

$\therefore \frac{dy}{dx} = 5 - 6x^2$

i.e., $m = 5 - 6x^2$,

where ‘m’ is the slope.

$\therefore \frac{dm}{dt} = -12x \frac{dx}{dt} = -12x\ (2) = -24x$

$\therefore \frac{dm}{dt}]_{x=3} = -24(3) = -72.$

Hence, the rate of the change of the slope = -72.

2. Solution:

Let x_1 and $x_2 \in R$.

Now $x_1 > x_2$

$\Rightarrow 7x_1 > 7x_2$

$\Rightarrow 7x_1 - 3 > 7x_2 - 3$

$\Rightarrow f(x_1) > f(x_2)$.

Hence, ‘f’ is strictly increasing function in R.

3. Solution:

We have: f(x) = $4x^3 - 18 \times 2 - 27x - 7$

$\therefore$ f(x) = $12x^2 - 36x + 27 = 12(x^2 - 3x) + 27$

= $12(x^2 - 3x + 9/4) + 27 - 27$

= $12(x - 3/2)^2 \forall$ x$\in$ R.

Hence, f(x) is always increasing in R.

4. Solution:

The given curve is $x - at^2$, y = 2at.

$\therefore \frac{dx}{dt}$ = 2at

$\frac{dx}{dt}$ = 2a

$\therefore \frac{dy}{dx} = \frac{\text{dy/dt}}{\text{dx/dt}} = \frac{2a}{2at} =$

Hence, slope of the tangent at t = 2 is: $\frac{dy}{dx}]_{t=2} = \frac{1}{2}$

5. Solution:

(i) We have:

f(x) = $(2x - 1)^2 + 3$.

Here Df = R.

Now f(x) $\geq$ 3.

[$\because (2x - 1)^2 \geq 0$ for all x $\in$ R]

However, maximum value does not exist.

[$\because$ f(x) can be made as large as we please]

(ii) We have:

f(x) = $16x^2 - 16x + 28$.

Here Df = R.

Now f(x) = 16 (x^2 – x + 14 + 24

= (16(x – $\frac{1}{2}$ $)^2$ + 24

⇒ f(x) ≥ 24.

[∵ 16(x – 12$)^2$ ≥ 0 for all x ∈ R

Hence, the minimum value is 24.

However, maximum value does not exist.

[∵ f(x) can be made as large as we please]

(iii) We have :

f(x) = – 1x + 11 + 3

⇒ f(x) ≤ 3.

[∵ -|x + 1| ≤ 0]

Hence, the maximum value = 3.

However, the minimum value does not exist.

[∵ f(x) can be made as small as we please]

(iv) We have :

f(x) = sin2x + 5.

Since – 1 ≤ sin 2x ≤ 1 for all x ∈ R,

– 1+5 ≤ sin2x + 5 ≤ 1+5 for all x∈ R

⇒ 4 ≤ sin2x + 5 ≤ 6 for all x ∈ R

⇒ 4 ≤ f(x) ≤ 6 for all x ∈ R.

Hence, the maximum value = 6 and minimum value = 4.

(v) We have :

f(x) = sin (sin x).

We know that – 1 ≤ sin x ≤ 1 for all x ∈ R

$\Rightarrow \sin(-1) \le \sin(\sin x) \le \sin 1$ for all $x \in R$

$\Rightarrow -\sin 1 \le f(x) \le \sin 1$.

Hence, maximum value = sin 1 and minimum value = -sin 1.

6. Solution:

The given curve is $x^2 = 2y$ ...(1)

Diff.w.r.t.t, $2x\frac{dx}{dt} = 2\frac{dy}{dt}$

$\Rightarrow 2x\frac{dx}{dt} = 2\frac{dx}{dt}$

$\because \frac{dy}{dt} = \frac{dx}{dt}$ given

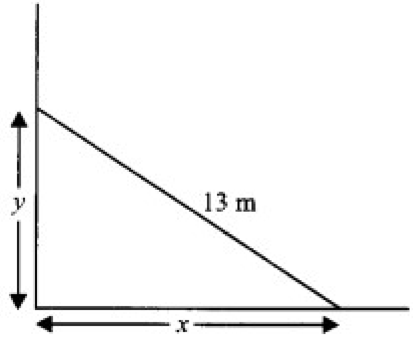
From (1), $1 = 2y \Rightarrow y = \frac{1}{2}$

Hence, the reqd. point is $(1, \frac{1}{2})$

Long Answer:

1. Solution:

Here, $\frac{dx}{dt} = 2$ cm/sec.



Now, $169 = x^2 + y^2$

$\Rightarrow \quad y = \sqrt{169 - x^2}.$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{169 - x^2}}(-2x)\frac{dx}{dt}$

$= -\frac{x}{\sqrt{169 - x^2}}(2)$

Hence, $\left.\frac{dy}{dt}\right]_{x=5} = \frac{-5}{\sqrt{169 - 25}}(2)$

$= \frac{-10}{12} = \frac{-5}{6}$ cm/sec.

Hence, the height is decreasing at the rate of 5/6 cm/sec.

2. Solution:

The given curves are:

$x^2 + y^2 = 4$ ....(1)

$(x - 2)^2 + y^2 = 4$ .... (2)

From (2),

$y = 4 - (x - 2)^2$

Putting in (1),

$x^2 + 4 - (x - 2)^2 = 4$

$\Rightarrow x^2 - (x - 2)^2 = 0$

$\Rightarrow (x + (x - 2)(x - x) + 2) = 0$

$\Rightarrow (2x - 2)(2) = 0$

⇒ x = 1.

Putting in (1),

$1 + y^2 = 4$

⇒ y = √3

∴ Point of intersection = (1, √3)

Diff. (1) w.r.t. x, $2x + 2y\frac{dy}{dx} = 0$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{[1,\sqrt{3}]} = -\frac{1}{\sqrt{3}} = m_1$$

Diff. (2) w.r.t. x, $2(x-2) + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{[1,\sqrt{3}]} = \frac{1}{\sqrt{3}} = m_2$$

$$\text{So, } \tan\theta = \left|\frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1+\left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)}\right| = \frac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1-\frac{1}{3}}$$

$$= \sqrt{3}.$$

Hence, $\theta = \frac{\pi}{3}$.

3. Solution:

Given function is:

$f(x) = -2x^3 - 9x^2 - 12x + 1.$

Diff. w.r.t. x,

$f'(x) = -6x^2 - 18x - 12$

$= -6(x + 1)(x + 2).$

Now, f'(x) – 0

⇒ x = -2, x = -1

⇒ Intervals are (-∞ – 2), (-2, -1) and (-1, ∞).

Getting f’ (x) > 0 in (-2, -1)

and f'(x) < 0 in (-∞, -2) u (-1, ∞)

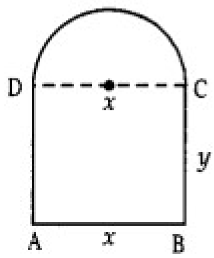
⇒ f(x) is strictly increasing in (-2, -1) and strictly decreasing in (-∞, 2) u (-1, ∞).

4. Solution:

Let ‘x’ and ‘y’ be the length and breadth of the rectangle ABCD.

Radius of the semi-circle = $\frac{x}{2}$.

Circumference of the semi-circle = $\frac{\pi x}{2}$



By the question, $x+2y+\frac{\pi x}{2} = 10$

$$\Rightarrow \quad 2x+4y+\pi x = 20$$

$$\Rightarrow \quad y = \frac{20-(2+\pi)x}{4} \quad \text{...(1)}$$

$$\therefore \text{ Area of the figure} = xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$= x\frac{20-(2+\pi)x}{4}+\pi\frac{x^2}{8}.$$

[*Using (1)*]

$$\text{Thus} \quad A(x) = \frac{20x-(2+\pi)x^2}{4}+\frac{\pi x^2}{8}.$$

$$\therefore \quad A'(x) = \frac{20-(2+\pi)(2x)}{4}+\frac{2\pi x}{8}$$

$$\text{and} \quad A''(x) = \frac{-(2+\pi)2}{4}+\frac{2\pi}{8}$$

$$= \frac{-4-2\pi+\pi}{4} = \frac{-4-\pi}{4}.$$

or Max ./Min. of A (x), A' (x) = 0

$$\frac{20-(2+\pi)(2x)}{4}+\frac{2\pi x}{8}=0$$

$20 - (2 + \pi)(2x) + \pi x = 0$

$20 + x(\pi - 4 - 2\pi) = 0$

$20 - x(4 + \pi) = 0$

$x = \frac{20}{4+\pi}$

and breadth $= y = \dfrac{20-(2+\pi)\cdot\frac{20}{4+\pi}}{4}$

$= \dfrac{80+20\pi-40-20\pi}{4(4+\pi)} = \dfrac{40}{4(4+\pi)} = \dfrac{10}{4+\pi}$.

And radius of semi-circle = $\frac{10}{4+\pi}$

Case Study Answers:

1. Answer :

i. (c) 2(x + y) = P

Solution:

Perimeter of floor = 2(Length + breadth)

⇒ P = 2(x + y)

ii. (c) $A = \frac{px - 2x^2}{2}$

Solution:

Area, A = length × breadth

⇒ A = xy

Since, P = 2(x + y)

$\Rightarrow \frac{P-2x}{2} = y$

$\therefore A = x\left(\frac{P-2x}{2}\right)$

$\Rightarrow A = \frac{Px-2x^2}{2}$

iii. (d) $\frac{P}{4}$

Solution:

We have, $A = \frac{1}{2}(Px - 2x^2)$

$\frac{dA}{dx} = \frac{1}{2}(P - 4x) = 0$

$\Rightarrow P - 4x = 0 \Rightarrow x = \frac{P}{4}$

Clearly, at $x = \frac{P}{4}, \frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ Area of maximum at $x = \frac{P}{4}$

iv. (c) $\frac{P}{4}$

Solution:

We have, $y = \frac{P-2x}{2} = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$

v. (a) $\frac{P^2}{16}$

Solution:

$$A = xy = \frac{P}{4} \cdot \frac{P}{4} = \frac{P^2}{16}$$

2. Answer :

i. (b) $x + 2y + \frac{\pi x}{2} = 10$

Solution:

Given, perimeter of window = 10m

∴ x + y + y + perimeter of semicircle = 10

$$\Rightarrow x + 2y + \pi \frac{2}{2} = 10$$

ii. (b) $A = 5x - \frac{x^2}{8} - \frac{\pi x^2}{8}$

Solution:

$$A = x \cdot y + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$= x\left(5 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{2}\frac{\pi x^2}{4}$$

$$[\therefore \text{From (i)}, y = 5 - \frac{x}{2} - \frac{\pi x}{4}]$$

$$= 5x - \frac{x}{2}^2 - \frac{\pi x^2}{4} + \frac{\pi x^2}{8} = 5x - \frac{x}{2}^2 - \frac{\pi x^2}{8}$$

iii. (c) $\frac{20}{4+\pi}$

Solution:

We have, $A = 5x - \frac{x^2}{2} - \frac{\pi r^2}{8}$

$\Rightarrow \frac{dA}{dx} = 5 - x - \frac{\pi x}{4}$

Now, $\Rightarrow \frac{dA}{dx} = 0$

$\Rightarrow 5 = x + \frac{\pi x}{4}$

$\Rightarrow x(4 + \pi) = 20$

$\Rightarrow x = \frac{20}{4+\pi}$

$$\left[\text{Clearly, } \frac{d^2A}{dx^2} < 0 \text{ at } x = \frac{20}{4+\pi}\right]$$

iv. (d) $\frac{50}{4+\pi}$

Solution:

At $x = \frac{20}{x} = \frac{20}{4+\pi}$

$$A = 5\left(\frac{20}{4+\pi}\right) - \left(\frac{20}{4+\pi}\right)^2 \frac{1}{2} - \frac{\pi}{8}\left(\frac{20}{4+\pi}\right)^2$$

$$= \frac{100}{4+\pi} - \frac{200}{(4+\pi)^2} - \frac{50\pi}{(4+\pi)^2}$$

$$\frac{(4+\pi)(100)-200-50\pi}{(4+\pi)^2} = \frac{400+100\pi-200-50\pi}{(4+\pi)^2}$$

$$\frac{200+50\pi}{(4+\pi)} = \frac{50(4+\pi)}{(4+\pi)} = \frac{50}{4+\pi}$$

v. (a) $\frac{10}{4+\pi}$

Solution:

We have, $y = 5 - \frac{x}{2} - \frac{\pi x}{4} = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$

$= 5 - x\left(\frac{2+\pi}{4}\right) = 5 - \left(\frac{20}{4+\pi}\right)\left(\frac{2+\pi}{4}\right)$

$= 5 - 5\frac{(2+\pi)}{4+\pi} = \frac{20+5\pi-10-5\pi}{4+\pi} = \frac{10}{4+\pi}$

Assertion and Reason Answers:

1. a) Both A and R are true and R is the correct explanation of A.

Solution:

Given that f(x)=2+cosx

Clearly f(x) is continuous and differentiable everywhere Also f′(x) = −sinx ⇒ f′(x=0)

⇒ −sinx = 0 ⇒ x = nπ

∴ These exists C ∈ [t, t+π] for t ∈ R

such that f′ (C) = 0

∴ Statement-1 is true Also

f(x) being periodic function of period 2π

∴ Statement-2 is true, but Statement-2 is not a correct explanation of Statement -1.

2. (a) Both A and R are true and R is the correct explanation of A.

Solution:

Given $f(x)=x^3-2x^2-1=0$

Here, $f(2)=(2)^3-2(2)^2-1=8-8-1=-1$

and $f(3)=(3)^3-2(3)^2-1=27-18-1=8$

∴ $f(2)f(3)=(-1)8=-8<0$

⇒ One root of f(x) lies between 2 and 3

∴ Given Assertion is true Also Reason R is true and valid reason

∴ Both A and R are correct and R is correct explanation of A.